Algebraic Topology

Algebraic Topology builds "functions" (actually functors)

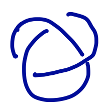
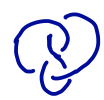
$$\left\{\begin{matrix}\text{Topological spaces,} \\ \text{continuous maps}\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}\text{algebraic things,} \\ \text{algebraic maps}\end{matrix}\right\}$$

e.g. groups, rings, vector spaces

the main point is to show two topological spaces are different

e.g. $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$

homeomorphic

$\mathbb{R}^3 -$  $\not\cong \mathbb{R}^3 -$ 

but can use alg. top. for many other things

1) maps between spaces

- does a given space M embed in N?

  e.g. for what m does $\mathbb{R}P^n$ embed in $\mathbb{R}^m$?

  (answer not known in general!)

- can you "lift" a map?

  i.e. given $f: A \to B$ and $\pi: E \to B$

  does there exist $\tilde{f}: A \to E$ s.t. $\pi \circ \tilde{f} = f$?

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

this includes $\exists$ of sections of bundles

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ B \arrow[u, dashed, bend right, "s"'] \end{tikzcd}$$

- Fixed points of maps

  eg. Brower fixed point theorem:

  every map $D^2 \to D^2$ has a fixed pt

2) Group actions

   eg Which finite groups act freely on S^n?

3) Group theory

   eg Every subgroup of a free group is free

   $[F_n, F_n]$ is not finitely generated $(n>1)$

   free group rank n

4) Algebra

   eg prove the fundamental theorem of algebra

In this course we develop

1) fundamental group $\pi_1(X)$

   and covering spaces

2) Homology groups $H_k(X)$ $\quad k = 0, 1, 2, \ldots$

3) Cohomology ring $H^*(X) = \oplus H^k(X)$

but before we start we will develop so important ideas that will be used throughout the course

0. Homotopy and CW Complexes

A. CW complexes

We develop alg. top. for all topological spaces, but a convenient (and very large) class of spaces to study are CW complexes

let $D^n \subset \mathbb{R}^n$ be the unit disk

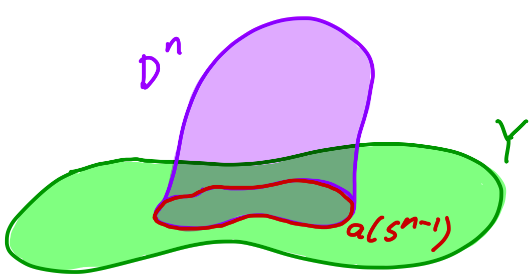
$S^{n-1} = \partial D^n$ its boundary

given
- Y a topological space and
- $a: S^{n-1} \to Y$ a continuous map

maps will be assumed to be continuous even if it is not stated

the space obtained from Y by attaching an n-cell (via a) is

$$Y \cup_a D^n = Y \amalg D^n / \{x \sim a(x)\}_{x \in S^{n-1}}$$



$Y \cup_a D^n$ is given the quotient topology

An n-complex, or n-dimentional CW complex is defined inductively by

a (-1) complex is $\varnothing$

an n-complex X^n is any space obtained from an $(n-1)$-complex X^{n-1} by attaching n-cells if $X = \bigcup_{n=0}^{\infty} X^n$, where X^n is an n-complex obtained by attaching n-cells to X^{n-1}

the we say X is an infinite dimensional complex

we say a CW complex is finite if it only involves a finite number of cells

the k-skeleton of X, is the union $X^{(k)}$ of all q-cells for $q \le k$

Remarks:

1) C in CW stands for closure finite and just means the closure of each cell is contained in the union of finitely many cells

2) W in CW stands for weak topology and means a set S in X is open $\Leftrightarrow$ $S \cap X^{(k)}$ open for all k

(this is automatic if X is finite dimensional)

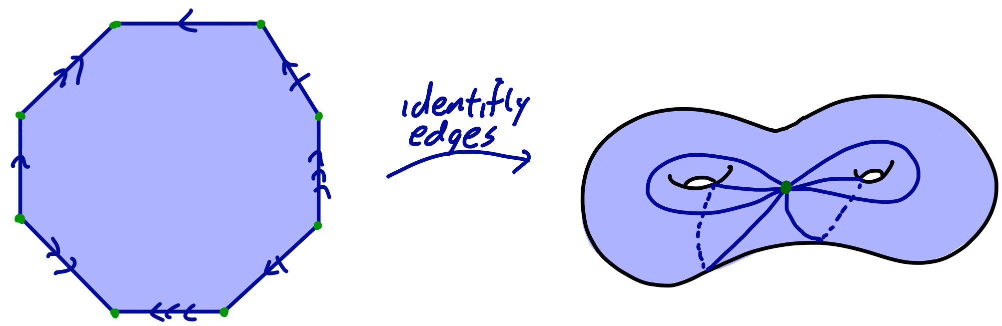
3) CW complexes are Hausdorff spaces (see Hatcher)

Exercise: Show the product of CW complexes is a CW complex.

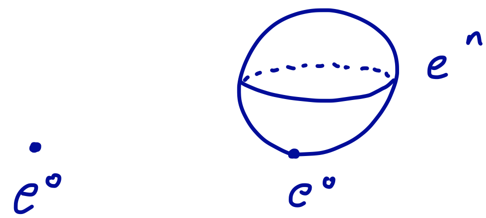
Examples:

1) 1-dim CW complexes are graphs

2) Surfaces are CW complexes



3) $S^n = e^0 \cup e^n$ $\quad$ e^i an i-cell

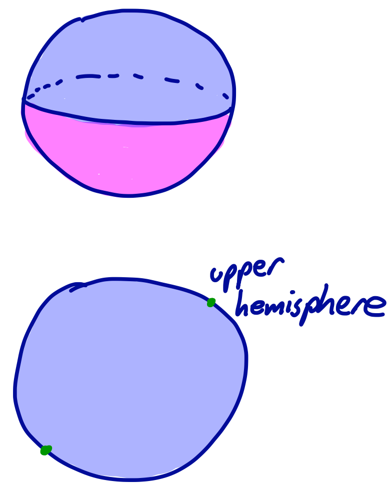


4) $\mathbb{R}P^n$ = lines in $\mathbb{R}^{n+1}$

= S^n with antipodes identified

= D^n with antipodes on ∂D^n identified

= $\mathbb{R}P^{n-1}$ with D^n attached



since $\mathbb{R}P^0 = \{pt\}$ we see inductively that $\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \ldots \cup e^n$ is a CW-complex

Exercise: Show $\mathbb{C}P^n$ is a CW complex

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \ldots \cup e^{2n}$$

5) (smooth) manifolds are CW complexes (see Hatcher)

a subcomplex of a CW complex X is a closed subspace $A \subset X$ that is a union of cells in X

note A is a CW complex too

(X, A) is called a CW pair

B Homotopy

A fundamental notion in algebraic topology is homotopy and homotopy equivalence let X and Y be topological spaces

two maps $f,g: X \to Y$ are homotopic, $f \sim g$, if there is a continuous map

$$\Phi: X \times [0,1] \longrightarrow Y$$

(Φ is called a homotopy)

s.t. $f(x) = \Phi(x,0) \quad \forall x \in X$

$g(x) = \Phi(x,1) \quad \forall x \in X$

Remarks:

1) Φ gives a family of maps $\phi_t: X \to Y$ where $\phi_t(x) = \Phi(x,t)$

these maps are "continuous in t" in the sence that Φ is continuous.

$\boxed{\text{so maps are homotopic if we can continuously deform one into the other}}$

2) if $A \subset X$, then we say the homotopy from f to g is relative to A, denoted $f \sim_A g$, if in addition to above

$$\Phi(x,t) = f(x) = g(x) \quad \forall x \in A,\ t \in [0,1]$$

3) if $A \subset X$ and $B \subset Y$, then the notation $f: (X,A) \to (Y,B)$ means $f: X \to Y$ is a map and $f(A) \subset B$

we say f is a map of pairs

if $f,g: (X,A) \to (Y,B)$, then they are homotopic (as maps of pairs) if $\exists$ a homotopy s.t. each ϕ_t is a map of pairs

Example: for any space X any map $f: X \to [0,1]$ is homotopic to the constant map $g(x) = 0$

the homotopy is $\Phi: X \times [0,1] \longrightarrow [0,1]$

$$(x,t) \longmapsto (1-t) f(x)$$

Exercise: homotopy is an equivalence relation on maps $X \to Y$

let $C(X,Y) = \{\text{continuous maps } X \to Y\}$

$$[X,Y] = C(X,Y)/\sim \quad \leftarrow \text{homotopy}$$
$$= \{\text{homotopy classes of maps } X \to Y\}$$

Examples:

1) for any X
$$[X,[0,1]] = \{g(x)=0\}$$

2) for any X
$$[\{*\},X] = \{\text{path components of } X\}$$
one point space

We call a space X pointed if it has a "base point" $x_0 \in X$
(just some prechosen fixed point)

given two pointed spaces $(X,x_0),(Y,y_0)$
$$[X,Y]_0 = \{\text{homotopy classes of maps of pairs } (X,\{x_0\}) \to (X,\{y_0\})\}$$

let y_0 be the north pole in the n-sphere S^n
(i.e. S^n = unit sphere in $\mathbb{R}^{n+1}$
$y_0 = (0,0,\dots 0,1)$)

the n^{th} homotopy group of a (pointed) space (X,x_0) is
$$\pi_n(X,x_0) = [S^n,X]_0$$

these are all groups and we will spend some time studying $\pi_1(X,x_0)$ which is also called the fundamental group

Question: For what Y is $[Y,X]_0$ "naturally" a group for all X? H-space
For what Y is $[X,Y]_0$ "naturally" a group for all X? H'-space

note: given a map $f: X_1 \to X_2$ there is a natural function

$$f_*: [Y, X_1] \to [Y, X_2] : g \mapsto f \circ g$$

and

$$f^*: [X_2, Y] \to [X_1, Y] : g \mapsto g \circ f$$

(Proof: just compose homotopy with f)

Rmk: Natural in question above means f_*, resp f^*, is a homomorphism

We say $f: X \to Y$ is the homotopy inverse of $g: Y \to X$ if $f \circ g \sim id_Y$ and $g \circ f \sim id_X$

if $g: Y \to X$ has a homotopy inverse then we say g is a homotopy equivalence and we say X and Y are homotopy equivalent or have the same homotopy type and write $X \simeq Y$

Exercise: This is an equivalence relation

lemma 1:

The following are equivalent

1) $X \simeq Y$
2) for any space Z there is a one-to-one correspondence

$$\phi_Z : [X, Z] \to [Y, Z]$$

such that for all continuous maps $h: Z \to Z'$

$$\begin{tikzcd} {[X, Z]} \arrow[r, "\phi_Z"] \arrow[d, "h_*"] \arrow[dr, phantom, "\circ"] & {[Y, Z]} \arrow[d, "h_*"] \\ {[X, Z']} \arrow[r, "\phi_{Z'}"] & {[Y, Z']} \end{tikzcd}$$

(commutes)

3) for any space Z there is a one-to-one correspondence

$$\phi^Z : [Z,X] \to [Z,Y]$$

such that for all continuous $h: Z \to Z'$

$$\begin{tikzcd}
[Z',X] \arrow[r, "\phi^{Z'}"] \arrow[d, "h^*"] \arrow[dr, phantom, "\circ"] & [Z',Y] \arrow[d, "h^*"] \\
[Z,X] \arrow[r, "\phi^Z"] & [Z,Y]
\end{tikzcd}$$

Proof: Exercise

Remark: So two spaces are homotopy equivalent iff homotopy classes of maps to and from the spaces are "naturally equivalent"

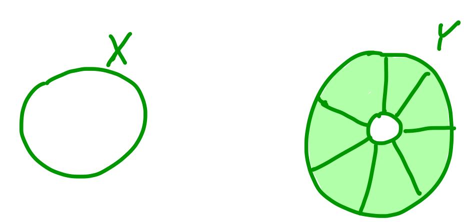
Examples:

0) if X and Y are homeomorphic, then they are homotopy equivalent.

1) $X = S^1$ is homotopy equivalent to $Y = S^1 \times [0,1]$

indeed: $f: X \to Y : x \mapsto (x,0)$

$g: Y \to X : (x,s) \mapsto x$



note: $g \circ f = id_X$

$f \circ g \sim id_Y$ by $\phi_t : Y \to Y : (x,s) \mapsto (x,ts)$

2) A space X is called contractible if it has the homotopy type of a point.

e.g. $\mathbb{R}^n \simeq \{*\}$ (exercise)

3) if $A \subset X$ then a retraction is a map $r: X \to A$ such that $r(x) = x \ \forall x \in A$

a deformation retraction of X to A is a homotopy, rel A, from the identity on X to a retraction:

$$\begin{aligned} &\phi_t : X \to X && t \in [0,1] \\ &\phi_0(x) = x && \forall x \in X \\ &\phi_1(X) \subset A && \\ &\phi_t(x) = x && \forall x \in A \text{ and } t \end{aligned}$$

note: If X deformation retracts to A then $X \simeq A$

indeed let

- ϕ_t be homotopy above
- $i: A \to X$ the inclusion map

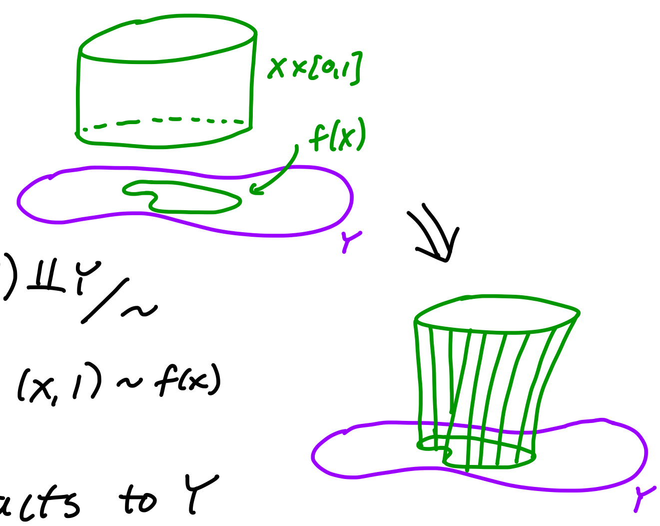
then i and ϕ_1 are homotopy inverses

since $\phi_1 \circ i = id_A$ and $i \circ \phi_1 = \phi_1 \sim \phi_0 = id_X$

given spaces X, Y and a map $f: X \to Y$ the mapping cylinder M_f is

$$M_f = (X \times [0,1]) \amalg Y /\sim$$

where $(x,1) \sim f(x)$



note: M_f deformation retracts to Y

indeed $\tilde{\phi}_t : \begin{array}{l} (x,s) \in X \times [0,1] \longmapsto (x, (1-t)s + t) \in X \times [0,1] \\ y \in Y \longmapsto y \in Y \end{array}$

induces maps $\phi_t : M_f \to M_f$ s.t. $\phi_0 = id_{M_f}$

$\phi_1(M_f) \subset Y$

$\phi_t(y) = y \quad \forall y \in Y$

thus $M_f \simeq Y$

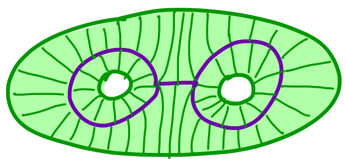
there are obvious inclusions $i: X \to M_f : x \mapsto (x,0)$

$j: Y \to M_f : y \mapsto y$ (j has homotopy inverse φ_1)

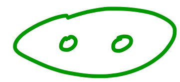
now
$$\begin{tikzcd} & Y \arrow[d, "\simeq" description, "j"] \\ X \arrow[ur, "f"] \arrow[r, "i"'] & M_f \end{tikzcd}$$
and $j \circ f \sim i$

Slogan: Any map is an inclusion upto homotopy

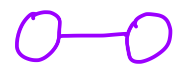
examples

$Z =$ 

let X = 3 green circles



let Y_1 = purple curves

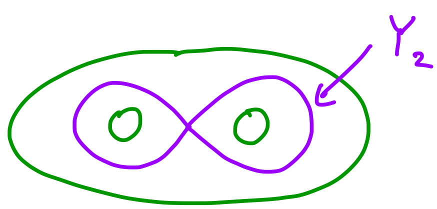


let $f: X \to Y_1$ given by following lines in picture

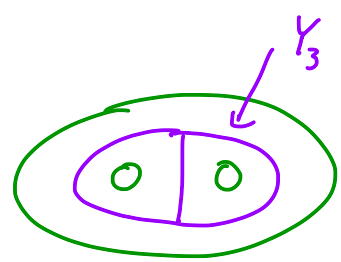
note Z is homeomorphic to M_f

so $Z \simeq Y_1$

similarly



$Z \simeq Y_2$



$Z \simeq Y_3$

So $Y_1 \simeq Y_2 \simeq Y_3$ even though it is not clear what the homotopy equivalence is!

Two criteria for homotopy equivalence

lemma 2:

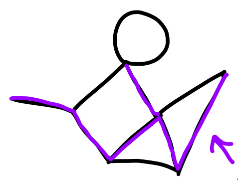
If (X,A) is a CW pair, and A is contractible then $X \simeq X/A$ ← collapse A to point

examples

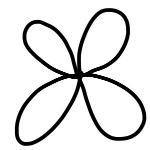
1) X a graph
A = any edge connecting distinct verticies $\Big\}\ X/A \simeq X$

$$\boxed{\text{so any connected graph is homotopy equivalent to a wedge of circles}}$$

X



$X/A \cong$

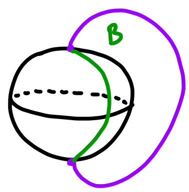


$\underbrace{S^1 \vee S^1 \vee \ldots \vee S^1}_{\text{wedge of } S^1\text{'s}}$

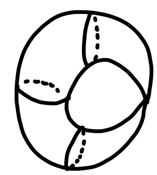
$(X, x_0) \vee (Y, y_0) = X \amalg Y/_{\{x_0 \sim y_0\}}$

2)

$X =$

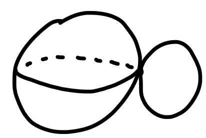


$X/A \cong$



$\cong S^2/\text{poles identified}$

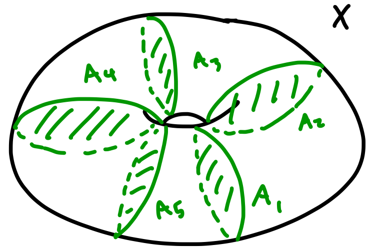
$X/B \cong$



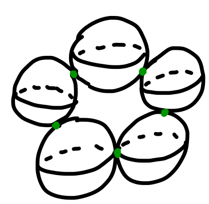
$\cong S^2 \vee S^1$

$X/A \simeq X \simeq X/B$

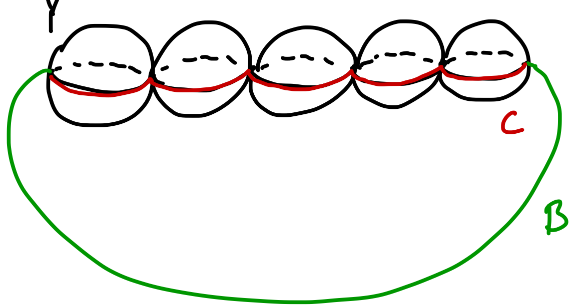
3)



$\xrightarrow{\simeq}$

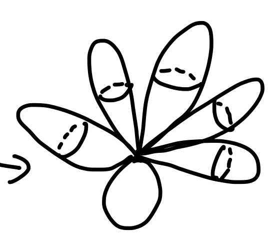


$\left(\left(X/A_1\right)/A_2\right)\ldots/A_5$



$\xrightarrow{\simeq}$ Y/B

$\xrightarrow{\simeq}$

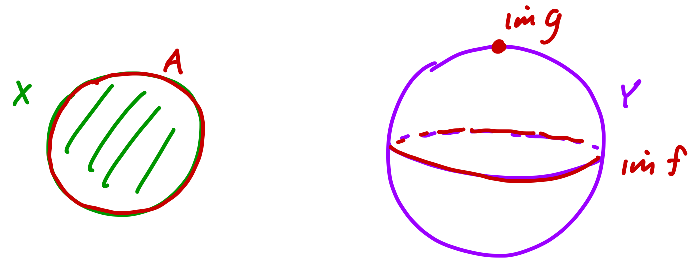


Y/C

lemma 3:

> let (X,A) be a CW pair
> $f, g : A \to Y$ homotopic maps
> Then $X \cup_f Y \simeq X \cup_g Y$

example:



$X = D^n \qquad A = \partial D^n$

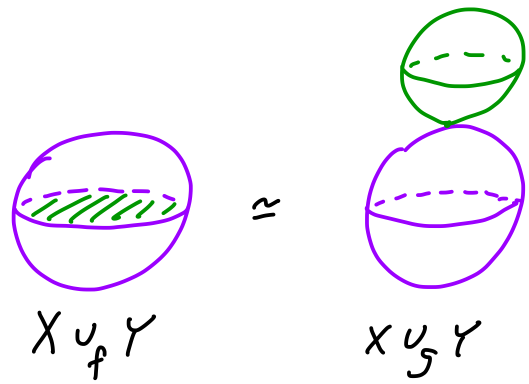
$Y = S^n$

$f : A \to Y$ map A to equator

$g : A \to Y$ constant map

exercise: $f \simeq g$

so $\quad X \cup_f Y \simeq X \cup_g Y$



To prove both lemmas we need the homotopy extention property (HEP)

A space X and a subspace $A \subset X$ has the HEP if whenever we have a map $F_0 : X \to Y$ and a homotopy $f_t : A \to Y$ of $f_0 = F_0|_A$ then we can extend the f_t to $F_t : X \to Y$

lemma 4:

> A pair (X, A) has the HEP $\Leftrightarrow$ $(X \times \{0\}) \cup (A \times [0,1])$ is a retract of $X \times [0,1]$

see Hatcher for general case

Proof: $(\Leftarrow)$ we assume A is closed (not nec. but makes proof easier and most examples satisfy this)

given the retract $r : X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$

and any map $F_0 : X \to Y$ and homotopy $f_t : A \to Y$ of $f_0 = F_0|_A$

note this defines a map $\quad F : (X \times \{0\}) \cup (A \times [0,1]) \to Y$

F is continuous since A is closed

now $F_0 : X\times[0,1] \to Y$ is the desired homotopy!

($\Rightarrow$) Consider the identity map $F : X\times\{0\}\cup A\times[0,1] \to X\times\{0\}\cup A\times[0,1]$

this gives $f_0 : X \to X\times\{0\}\cup A\times[0,1]$ by $F|_X$

and $f_t : A \to X\times\{0\}\cup A\times[0,1]$ by $f_t = F|_{A\times\{t\}}$

so HEP $\Rightarrow \exists\ F_t : X \to X\times\{0\}\cup A\times[0,1]$

the F_t give a map $r : X\times[0,1] \to X\times\{0\}\cup A\times[0,1]$
$(x,t) \longmapsto F_t(x)$

that is clearly a retraction

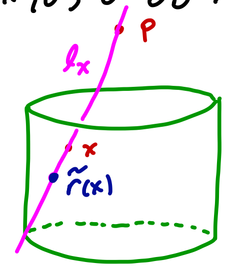
lemma 5:

If (X,A) is a CW pair, then $X\times\{0\}\cup A\times[0,1]$ is a (deformation) retract of $X\times[0,1]$

In particular, (X,A) has the HEP

Proof:

Main point: for any disk D^n there is a deformation retraction of $D^n\times[0,1]$ to $D^n\times\{0\}\cup\partial D^n\times[0,1]$



Pf: let $D^n \subset \mathbb{R}^n = \mathbb{R}^n\times\{0\} \subset \mathbb{R}^{n+1}$

so $D^n\times[0,1] \subset \mathbb{R}^{n+1}$

let $p = (0,0,\dots,0,2)$

given $x \in D^n\times[0,1]$ let l_x = line through x and p

and set $\tilde r(x) = l_x \cap (D^n\times\{0\}\cup\partial D^n\times[0,1])$

unique point!

clear $\tilde r$ is a retraction (need to check continuous exercise)

and $\tilde r_t = t\tilde r + (1-t)\,\mathrm{Id}_{D^n\times[0,1]}$

is a deformation retraction

we define r on $X^{(0)} \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ as follows

if a vertex $D^0 \subset A$, then let r be the identity on $D^0 \times [0,1]$

if $D^0 \not\subset A$, then let r send any point in $D^0 \times [0,1]$ to D^0 in $X \times \{0\}$

now inductively assume we have defined r on the $(k-1)$ skeleton of X, that is $X^{(k-1)} \times [0,1] \to X \times \{0\} \cup A \times [0,1]$

for each k-cell D^k of X

if $D^k \subset A$ then let r be the identity map on $D^k \times [0,1]$

if D^k is <u>not</u> a cell in A then note $\partial D^k \times [0,1] \to X^{(k-1)} \times [0,1]$ where r is already defined

and we have an "inclusion"

$$\begin{tikzcd} D^n \arrow[r, "i"] \arrow[rr, bend right, "j"'] & X^{(n-1)} \cup D^n \arrow[r, "q"] & X^{(n-1)} \cup D^n / (x \in \partial D^n) \sim a(x) \in X^{(n-1)} \end{tikzcd}$$

where $a : \partial D^n \to X^{(n-1)}$ is the attaching map for D^n

so we have a map $D^n \times \{0\} \xrightarrow{j} X \times \{0\}$

i.e. r is defined on $(D^n \times \{0\}) \cup (\partial D^n \times [0,1])$

so composing $\tilde{r}$ above with the above maps extends r over $D^n \times [0,1]$

and eventually all of $X^{(n)} \times [0,1]$ ▦

<u>Proof of lemma 2</u>: Actually we show for any pair (X, A) satisfying HEP with A contractible, the quotient map $q : X \to X/A$ is a homotopy equivalence

for this note there is a homotopy $f_t : A \to A \subset X$ st. $f_0 = id_A$, $f_1 = $ constant map

note $f_0 = F_0|_A$ where $F_0 = id_X$

so HEP gives a homotopy $F_t : X \to X$ extending f_t

since $F_t(A) \subset A$ for all t we get maps $\bar{F}_t : X/A \to X/A$

$$\begin{tikzcd} X \arrow[r, "F_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{F}_t"'] & X/A \end{tikzcd}$$

also $F_1(A) = pt$ so F_1 also gives a map $h : X/A \to X$

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "h"] \arrow[r, "\bar{F}_1"'] & X/A \end{tikzcd}$$

you can easily check $h \circ q = F_1$ and $q \circ h = \bar{F}_1$

but now $h \circ q = F_1 \sim F_0 = id_X$

$q \circ h = \bar{F}_1 \sim \bar{F}_0 = id_{X/A}$

Proof of lemma 3:

Recall we have (X, A) and maps $f, g : A \to Y$

that are homotopic

let $F : A \times [0,1] \to Y$ be the homotopy

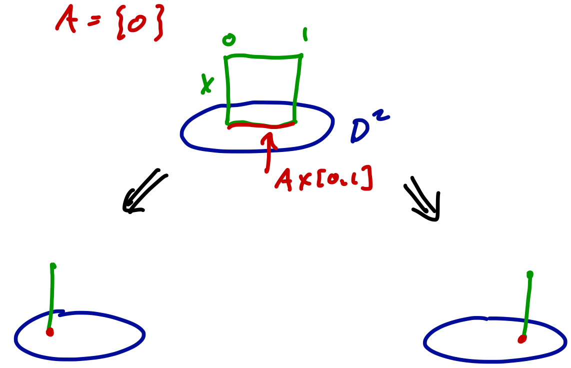
now let $M_F = X \times [0,1] \cup_F Y$

claim M_F deformation retracts to $X \cup_f Y$ and $X \cup_g Y$

$\therefore X \cup_f Y \simeq X \cup_g Y$

e.g. $X = [0,1]$ $\quad Y = D^2$

$A = \{0\}$



from lemma 5 we have a deformation retraction of $X \times [0,1]$ to $X \times \{0\} \cup A \times [0,1]$

exercise: $(X \times \{0\} \cup A \times [0,1]) \cup_F Y$

$$\cong$$

$X \cup_f Y$

given this we see the above deformation retraction induces a deformation retraction of M_F to $X \cup_f Y$

Proof of lemma 5 also shows $X \times [0,1]$ deformation retracts onto $X \times \{1\} \cup A \times [0,1]$

<u>exercise:</u> $(X \times \{1\} \cup A \times [0,1]) \cup_F Y \cong X \cup_g Y$

so as above $M_F \simeq X \cup_g Y$